J. Fluid Mech. (1967), *vol.* 29, *part* 2, *pp.* 209–231
Printed in Great Britain

Effect of viscosity on gravity waves and the upper boundary condition

By MICHAEL YANOWITCH
Adelphi University, Garden City, New York

(Received 15 June 1966 and in revised form 20 December 1966)

The linearized problem of two-dimensional gravity waves in a viscous incompressible stratified fluid occupying the upper half-space $z > 0$ is investigated. It is assumed that the dynamic viscosity coefficient μ is constant and that the density distribution $\rho(z)$ is exponential. This leads to a fourth-order differential equation in the z co-ordinate, the coefficients of which depend on $\rho(z)$ and on a dimensionless parameter ϵ which is proportional to μ/σ, σ being the frequency of the oscillation. The problem is solved for small ϵ. It is found that there is a region in which the solutions behave like certain solutions of the inviscid problem (with $\epsilon = 0$). However, when the solutions of the inviscid problem are wave-like in z, they do not satisfy the radiation condition. This is because the viscosity, in addition to damping the motion for large z, reflects waves. The appropriate solution of the inviscid problem consists, therefore, of an incident and a reflected wave. As $\mu \to 0$, the ratio of the amplitudes of the reflected and the incident wave approaches $\exp(-2\pi^2 H/\lambda)$, where λ is the vertical wavelength, and H the density scale height. The solution, however, does not have a limit since the reflecting layer shifts, altering the phase of the reflected wave. The results of the analysis are supplemented by a number of numerically computed solutions, which are then used to discuss the validity of the linearization.

1. Introduction

The problem of waves in an ideal compressible fluid in a half-space, with a specified vertical temperature profile, has been investigated frequently, principally because of its connexion with the theory of atmospheric waves (see e.g. Lamb 1932; Eckart 1960). One of the difficulties encountered in this fluid model has been the question of how to formulate the so-called 'upper boundary condition'. This paper is concerned with a much simpler problem which exhibits the same difficulty. The problem is that of waves in an *incompressible* but stratified fluid, having a density distribution $\rho(z)$ which decreases exponentially with altitude z. The fluid is assumed to be *viscous*, and the 'upper boundary condition' is obtained by examining the solution when the viscosity is small. The results obtained this way differ considerably from those which would be obtained if viscosity were neglected at the outset.

In the linearized version of the more general problem for a compressible inviscid fluid the variables are separable, and one obtains an ordinary differential equation of second order in the vertical co-ordinate, z. Two further conditions are then required to determine a unique solution in the problem of forced oscillations, or an eigenfunction in the free oscillation problem. One of these is the well determined boundary condition on the vertical velocity at the ground, while the other condition depends on the assumptions about the vertical temperature distribution as $z \to \infty$, and on the parameters of the wave motion. As an illustration, let us consider the case of an isothermal atmosphere. Then, for certain values of the horizontal wave-number k and frequency σ, the differential equation has one solution with finite kinetic energy in a column of fluid, and one with infinite kinetic energy. In this case it is natural to impose as a second condition the finiteness of the kinetic energy. The resulting solution represents a horizontally propagating wave with a certain amplitude profile in the z co-ordinate:

$$f(z) \exp [i(kx - \sigma t)].$$

For other values of the parameters one can find two solutions which represent waves with oblique lines of constant phase:

$$g(z) \exp [i(kx - \sigma t \pm \beta z)]$$

(k, σ and β are all positive). Neither one of these has finite kinetic energy and there arises the problem, therefore, of imposing a condition (the upper boundary condition) which will select the correct linear combination of these wave-like solutions.

In the problem of atmospheric tides,† Pekeris (1937) derived a condition by considering the limiting case of a small artificial thermal conductivity. An objection to his results was raised by Weekes & Wilkes (1947), who pointed out that it leads sometimes to solutions which would require energy to be supplied from ∞. The requirement that no energy should come in from ∞ is, however, not sufficient to determine a unique solution, since any linear combination of the form

$$A \exp [i(kx - \sigma t - \beta z)] + B \exp [i(kx - \sigma t + \beta z)] \quad \text{with} \quad |B| < |A|$$

represents a wave with upward propagating energy. The choice made by Weekes & Wilkes of the solution with downward phase propagation ($B = 0$) implies, therefore, not only that the energy flux is upward, but that, in addition, there is no reflexion (the term multiplied by B being the reflected wave). This condition of upward energy flux and the absence of reflexion, to which we will refer as the 'radiation condition', has recently been questioned by Siebert (1961). Siebert, based on his observation that the resonance magnification curves are not smooth in the neighbourhood of the transition from horizontal to oblique waves, (as the horizontal wave-number is varied) has suggested that the solution should behave like a standing wave. It should be noted that precisely at the transition point none of the solutions are wave-like in the z-co-ordinate, and that all of them have infinite kinetic energy. Thus, none of the previously discussed conditions are applicable.

The related problem of mountain waves gives rise to a similar question. For this problem, the radiation condition was first derived by Lyra (1943) by making

† Although the tidal problem is formulated for an atmosphere on a rotating earth, the vertical equation is identical with the one for the half-plane.

use of artificial (Rayleigh) viscosity. The condition was rederived (for an isothermal atmosphere) later by considering a time dependent problem and showing that the solution which satisfies the radiation condition is ultimately established as $t \to \infty$ (Palm 1953; Wurtele 1953; Crapper 1958). However, even this approach has not avoided some controversy (e.g. Scorer 1958; Palm 1958).

Various other conditions are sometimes employed in different problems, the most notable, perhaps, being the condition that $dp/dt \to 0$ (where p is the pressure) (Eliassen 1948). This condition is used in problems formulated in pressure coordinates when the hydrostatic approximation is valid. However, in the tidal problem this condition is ineffective because it is automatically satisfied by all solutions of the differential equation.

It appears that the radiation condition is the most satisfactory upper boundary condition consistent with the linearized ideal fluid model, since it can be derived from a time dependent problem. However, this linear problem has a basic flaw in that it leads to solutions which violate the assumptions underlying the linearization. As is well known, the velocities increase exponentially with z for some solutions, and, what is perhaps worse, the same is true of the ratio of the density oscillation to the equilibrium density, and of the pressure oscillation to the equilibrium pressure. In view of the inconsistency of the solution with the original assumptions it seems desirable to re-examine the problem.

Perhaps the simplest approach is to retain the linearized formulation but to abandon the ideal fluid model by including viscosity. One expects the viscosity to play an increasingly important role at high altitudes since the density in the atmosphere decreases very rapidly (by a factor of 10^6 in the lowest 100 km) while the viscosity varies only slightly. Thus, the viscosity may be expected to damp the motion for large z, and in this way to justify the linearized formulation for sufficiently small disturbances. By taking the limit of the solutions of the viscous problem as the viscosity coefficient μ tends to zero one can hope to obtain an upper boundary condition which could be used with the inviscid problem.

In this paper we have selected the simplest viscous fluid problem which still retains the two characteristic features: an infinite domain, and an equilibrium density distribution which vanishes as $z \to \infty$, as a consequence of which the viscosity is expected to have a substantial effect of large z. The main simplifications, viz. the assumptions that the fluid is incompressible, has a constant viscosity coefficient μ, and a density decreasing exponentially with z, are introduced to make the problem analytically tractable. This model, although much too simple for a direct comparison with atmospheric motions, nevertheless has features which are relevant (this point will be discussed further in the introduction and in §10).

Small damping effects of viscosity on upward and downward propagating waves in a compressible isothermal atmosphere were investigated by Pitteway & Hines (1963). Other investigations have been restricted to the case of constant kinematic viscosity (e.g. Golitsyn 1965). This leads to a problem with constant coefficients, but the effect of viscosity is, of course, negligible for large z.

The viscous problem is formulated in §2 in terms of a stream function. This leads to a singular perturbation problem for a fourth-order differential equation with coefficients depending on the equilibrium density ρ and on a small dimensionless parameter ϵ which is proportional to μ/σ. To determine a solution uniquely one must impose the usual no slip condition at the ground, and, in addition, a condition that the rate of energy dissipation in a column of fluid should be finite. By introducing a new independent variable, essentially $\sqrt{(\rho/\epsilon)}$, the problem can be reformulated so that the solution depends on this variable only (§3). The required solution is then found in the form of a series (§5), the asymptotic behaviour of which yields the conclusions concerning the upper boundary condition (§§6, 7 and 8). The results of some numerical computations of solutions for various values of wave parameters are given in §10. These are used to estimate the magnitude of the excitation for which the linearized formulation can be expected to be valid.

The main results is that the radiation condition does *not* hold when the ratio of the vertical wavelength λ to the scale height H is large. For sufficiently small μ there is an 'inviscid region' in which the solution of the viscous problem can be approximated by an appropriate solution of the inviscid problem, which consists of an incident wave (with outgoing energy) and a reflected wave. The nature of the solution in this region is determined primarily by a relatively thin layer above the inviscid region. Above this reflecting layer the viscous forces predominate and the velocities decay to zero. As $\mu \to 0$ the reflecting layer recedes to ∞, but the ratio of the amplitudes of the reflected to the incident wave, $|B/A|$, tends to a limit, which is $\exp(-2\pi^2 H/\lambda)$. Thus, the ratio of the reflected to incident energy tends to a limit also. The solution, however, does *not* approach a limit at fixed values of z, for the shift of the reflecting layer causes a change in the relative phase between the two waves.

Since the limiting value of the magnitude of the reflexion coefficient $|B/A|$ is $\exp(-2\pi^2 H/\lambda)$, the solution behaves more and more like a standing wave as $\lambda \to \infty$, i.e. near the transition from horizontally propagating to oblique waves. In this region, therefore, Siebert's suggestion appears to be justified. As λ decreases, however, the amplitude of the reflected wave decreases rapidly, and when the dimensionless wave-number $\beta = 2\pi H/\lambda > 1$, the radiation condition is substantially correct. This situation is not too surprising, since the variation of the kinematic viscosity in a vertical wavelength is large when β is small, and can be expected to cause reflexion. However, it might be noted that a large change in the kinematic viscosity is required to produce a sizable reflexion, since even for $\beta = 1$, μ/ρ changes by a factor of about 400 in one vertical wavelength.

Reflexion due to viscosity will also take place in a compressible isothermal atmosphere (a special case is solved in Yanowitch 1967). The vertical equation for this model is very similar to the one for an incompressible fluid, the main difference being that for a compressible fluid there is a high frequency and a low frequency range of waves with oblique lines of constant phase, while for an incompressible fluid there is only a low frequency range. Reflexion due to viscosity can thus be expected to occur in the neighbourhood of both the high and low frequency transition points, and conclusions derived from the incompressible model are probably qualitatively correct for the low frequency range. The effect of viscosity is, therefore, likely to have some importance in the problem of semidiurnal oscillations. This is indicated by the fact that the second equivalent height for an atmosphere with an isothermal top layer is usually computed (using the radiation condition) to lie near the value at which the transition from horizontal to oblique wave propagation takes place (Wilkes 1949; Jacchia & Kopal 1951). Reflexion of waves may also be significant in the problem of propagation of waves due to large disturbances, since the wave components beyond cut-off may not be negligible.

2. Formulation of the problem

Suppose an incompressible, viscous, inhomogeneous fluid occupies the half-space $z > 0$ when at rest. It will be assumed that the dynamic viscosity coefficient μ is constant and that the equilibrium density distribution ρ depends on the altitude z only. (Later we will specialize to the case of exponentially decreasing density.) The linearized equations of motion are then†

$$\rho u_t + \tilde{p}_x = \mu \Delta u, \tag{1}$$

$$\rho w_t + \tilde{p}_z + g\tilde{\rho} = \mu \Delta w, \tag{2}$$

$$u_x + w_z = 0, \tag{3}$$

$$\tilde{\rho}_t + \rho' w = 0, \tag{4}$$

where u and w are the horizontal and vertical velocity components, $\tilde{p}$ and $\tilde{\rho}$ are the pressure and the density perturbations, g is the acceleration of gravity, Δ represents the two-dimensional Laplace operator and a prime denotes differentiation with respect to z. We will consider two-dimensional infinitesimal waves excited, for example, by a small oscillation of the lower boundary:‡ $z = d\,e^{i(kx-\sigma t)}$. Let Ψ be a stream function defined by $\Psi_z = u$, $\Psi_x = -w$. Then Ψ satisfies the differential equation

$$[\rho \Psi_{xx} + (\rho \Psi_z)_z]_{tt} - g\rho' \Psi_{xx} = \mu \Delta\Delta \Psi_t, \tag{5}$$

which can be obtained by eliminating the pressure and density perturbations. Letting $\Psi(x,z,t) = Z(z)\,e^{i(kx-\sigma t)}$, one obtains an ordinary differential equation of fourth order for $Z(z)$:

$$i\sigma^{-1}\mu[d^2/dz^2 - k^2]^2 Z - [(\rho Z')' - k^2\rho Z - g(k/\sigma)^2 \rho' Z] = 0.\S \tag{6}$$

The no-slip condition at the lower boundary requires that $u = 0$ and $w = -i\sigma d\,e^{i(kx-\sigma t)}$ at $z = 0$, i.e.

$$Z(0) = d\sigma/k, \quad Z'(0) = 0 \tag{7}$$

when $k \neq 0$, or when k and σ both tend to zero, with σ/k approaching a finite limit.

It is convenient to rewrite the problem in dimensionless form. Let H be a characteristic length related to the density stratification, for which one can take a suitable value of $(-\rho'/\rho)^{-1}$. Then, referring all lengths to H, velocities to $\sqrt{(gH)}$,

† A list of symbols appears at the end of the paper.

‡ The exact nature of the boundary condition is unimportant, as will be seen later.

§ If σ is replaced by $i\sigma$ one obtains an equation considered in stability theory (Chandrasekhar 1961, p. 430).

and the time to the period $\sqrt{(H/g)}$ (this corresponds to the Brunt–Väisälä frequency for this case), we can define

$$\rho^* = \rho/\rho(0), \quad z^* = z/H, \quad x^* = x/H, \quad k^* = kH, \quad \sigma^* = (H/g)^{\frac{1}{2}}\sigma,$$
$$Z^* = Z/(gH^3)^{\frac{1}{2}}, \quad \epsilon = \mu[4\rho(0)\,H^2\sigma]^{-1}. \tag{8}$$

Apart from the exponential factor, Z^* is a dimensionless stream function, with dZ^*/dz^* and $-ik^*Z^*$ being dimensionless horizontal and vertical velocity components. Frequently, instead of the parameter ϵ (which is proportional to the ratio of the Strouhal number to the Reynolds number) it will be more convenient to employ $\delta = -\ln \epsilon$.

Since everything will be expressed in terms of the dimensionless quantities from now on, the asterisk can be omitted without danger of confusion. Then, the differential equation (6) and the boundary conditions (7) can be rewritten in the form

$$4i\epsilon[d^2/dz^2 - k^2]^2\, Z - [(\rho Z')' - k^2\rho Z - (k/\sigma)^2\rho' Z] = 0, \tag{9}$$
$$Z(0) = d\sigma/Hk, \quad Z'(0) = 0. \tag{10}$$

If $\rho = e^{-z}$ (i.e. $\rho = \rho(0)\,e^{-z/H}$ in dimensional quantities), then (9) becomes

$$4[d^2/dz^2 - k^2]^2\, Z - e^{-(z-\delta+\frac{1}{2}i\pi)}[Z'' - Z' + rZ] = 0, \tag{11}$$

where $r = k^2(\sigma^{-2} - 1)$.

The problem for the differential equation (9) with boundary condition (10) is, of course, incomplete. Further conditions are needed to determine a unique solution. We will impose the following additional condition, which we will call the *dissipation condition* (DC): *the average rate (per period) of energy dissipation in an infinite vertical column of fluid* ($0 < z < \infty$) *of unit cross-section area must be finite.* The necessity of this condition is evident on physical grounds, since the rate at which work is done by a portion of the lower boundary of unit area on the fluid above is finite. It will be shown that for the case to be discussed this condition is also sufficient for uniqueness.† The DC appears to be a reasonable one, therefore, since any other solution of the differential equation (9) satisfying the boundary condition (10) requires an infinite energy flux from infinity.

To apply the DC we need only note that the dissipation function depends on the squares of the space derivatives of u and w. Consequently, the DC is equivalent to

$$\int_0^\infty |Z|^2 dz < \infty, \quad \int_0^\infty |Z'|^2 dz < \infty, \quad \int_0^\infty |Z''|^2 dz < \infty. \tag{12}$$

The complete energy flow equation for the differential equation (9) can be obtained by multiplying (9) by $\bar{Z}$ (the complex conjugate of Z) and integrating between $z = z_1$ and $z = z_2$ yields

$$4i\epsilon[\bar{Z}Z''' - \bar{Z}'Z'' - 2k^2\bar{Z}Z']_{z_1}^{z_2} - [\rho\bar{Z}Z']_{z_1}^{z_2} + \int_{z_1}^{z_2}\{\rho|Z'|^2 + k^2\rho|Z|^2 + k^2\sigma^{-2}\rho'|Z|^2\}dz$$
$$+ 4i\epsilon\int_{z_1}^{z_2}\{|Z''|^2 + 2k^2|Z'|^2 + k^4|Z|^2\}dz = 0. \tag{13}$$

† For the case of horizontal motion independent of x, one can show that the DC is correct for any density distribution.

The imaginary part of this expression is the required energy flow equation:

$$4\epsilon\,\mathrm{Re}\,[\bar{Z}Z''' - \bar{Z}'Z'' - 2k^2\bar{Z}Z']_{z_1}^{z_2} - \mathrm{Im}\,[\rho\bar{Z}Z']_{z_1}^{z_2} + 4\epsilon\int_{z_1}^{z_2}\{|Z''|^2 + 2k^2|Z'|^2 + k^4|Z|^2\}dz = 0. \quad (14)$$

The integral in (14) is proportional to the average rate of energy dissipation in a column of fluid of unit cross-section between z_1 and z_2, while the other terms are proportional to the fluxes of energy due to the work done by the pressure and the viscous forces at the ends of the column. The DC applied to (14) implies (12).

3. Heuristic discussion of the problem

We will be interested in the problem when the viscosity is small, i.e. for small values of ϵ. For any $\epsilon \neq 0$, no matter how small, the viscous terms will dominate for large z if $\rho(z) \to 0$ sufficiently fast as $z \to \infty$. Thus, as $z \to \infty$ it is expected that solutions of (9) will behave like solutions of

$$(d^2/dz^2 - k^2)^2 Z = 0, \quad (15)$$

i.e. like linear combinations of e^{-kz}, ze^{-kz}, e^{kz} and ze^{kz}, if $k \neq 0$ (for $\rho = e^{-z}$ this will be proved later). For convenience, we will always take k to be non-negative. Then, only the first two of the above solutions satisfy the DC and we are led, therefore, to the problem of finding a solution of (9) which satisfies the boundary conditions (10) and behaves like $(az+b)e^{-kz}$ as $z \to \infty$. For the case $k = 0$, the solutions of (15) are polynomials of the third degree and the DC requires that $Z \sim az+b$ as $z \to \infty$.

For small ϵ the solution to the problem can now be expected to behave in the following way. There will be an 'inviscid region' (IR) in which ρ/ϵ is large and the solution can be approximated by some solution Z_i of the inviscid problem,

$$(\rho Z_i')' - k^2\rho Z_i - k^2\sigma^{-2}\rho' Z_i = 0, \quad Z_i(0) = 1. \quad (16)$$

This region is connected to the lower boundary $z = 0$ by a boundary layer (BL) which reduces Z' (i.e. the horizontal velocity) to zero as $z \to 0$. In the BL Z will behave approximately like a solution to $4i\epsilon Z^{(\mathrm{IV})} - Z'' = 0$, so that the width of the BL will be of the order of $\epsilon^{\frac{1}{2}}$. For sufficiently large z, i.e. for sufficiently small ρ/ϵ, the solution will be determined by the viscous forces and the pressure forces ('viscous region' – VR). The IR and the VR will be joined in a transition region (TR) in which ρ/ϵ changes from large to small values and both the viscous and the non-viscous terms in the differential equation will be significant. The main problem, of course, is to connect the solutions across the TR.

Once a solution has been found, its behaviour in the IR can be investigated as $\epsilon \to 0$ in order to determine the correct 'upper boundary condition' for the inviscid problem. When $\rho = e^{-z}$ the differential equation for the inviscid problem reduces to one with constant coefficients:

$$Z_i'' - Z_i' + rZ_i = 0, \quad (17)$$

and the form of the solution depends on whether $(\frac{1}{4}-r)$ is positive, negative or zero. Letting $\alpha = (\frac{1}{4}-r)^{\frac{1}{2}}$ when $r < \frac{1}{4}$, and $\beta = (r-\frac{1}{4})^{\frac{1}{2}}$ when $r > \frac{1}{4}$ $(\alpha > 0, \beta > 0)$, we can express solutions in the following form:

$$\left.\begin{aligned} Z_i &= e^{\frac{1}{2}z}[A\,e^{-\alpha z}+B\,e^{\alpha z}] \quad &\text{if} \quad r < \tfrac{1}{4}, \\ &= e^{\frac{1}{2}z}[A\,e^{-i\beta z}+B\,e^{i\beta z}] \quad &\text{if} \quad r > \tfrac{1}{4}, \\ &= e^{\frac{1}{2}z}[A+B\ln z] \quad &\text{if} \quad r = \tfrac{1}{4}, \end{aligned}\right\} \tag{18}$$

(here and elsewhere A and B denote constants which can assume different values in different expressions). The problem of determining the 'upper boundary condition' is then equivalent to that of finding the ratio B/A when $r > \frac{1}{4}$ and ϵ is sufficiently small. It will be seen later that in this case the solution of the complete problem, Z, can be approximated by Z_i, but that A and B vary with ϵ and B/A does not tend to a limit as $\epsilon \to 0$.

4. Change of variables and asymptotic behaviour

From this point on we will consider an exponential density distribution, i.e. $\rho = e^{-z}$. Since the solution will depend on ρ/ϵ, it will be convenient to introduce a new variable, $\xi = \exp[-\frac{1}{2}(z-\delta+\frac{1}{2}i\pi)]$. Then d/dz goes over into $-\frac{1}{2}\Theta$, where Θ represents the operator $\xi d/d\xi$, and the differential equation (11) transforms into

$$LZ = \{(\Theta^2-4k^2)^2-\xi^2(\Theta^2+2\Theta+4r)\}Z = 0. \tag{19}$$

Real values of z go over into points on the ray $\arg\xi = \frac{1}{4}\pi$ in the complex ξ-plane, with $z = \infty$ being mapped into $\xi = 0$ and $z = 0$ into $\xi = \xi_1 = \exp\frac{1}{2}(\delta-\frac{1}{2}i\pi)$. The boundary condition (10) becomes

$$Z(\xi_1) = d\sigma/Hk, \quad \Theta Z(\xi_1) = 0. \tag{20}$$

The differential equation for the inviscid problem in the new variable is now

$$(\Theta^2+2\Theta+4r)Z_i = 0, \tag{21}$$

and solutions can be written in the form

$$\left.\begin{aligned} Z_i &= \xi^{-1}(A\xi^{2\gamma}+B\xi^{-2\gamma}) \quad &(r \neq \tfrac{1}{4}), \\ &= \xi^{-1}(A+B\ln\xi) \quad &(r = \tfrac{1}{4}). \end{aligned}\right\} \tag{22}$$

Here $\gamma = (\frac{1}{4}-r)^{\frac{1}{2}} = \alpha > 0$ if $r < \frac{1}{4}$, and $\gamma = i\beta$, $\beta = (r-\frac{1}{4})^{\frac{1}{2}} > 0$ if $r > \frac{1}{4}$, while ξ^a denotes $e^{a\ln\xi}$ as usual, where $\ln\xi$ is the principal branch of the logarithm.

The point $\xi = 0$ is a regular singular point of (19), and in the neighbourhood of this point solutions behave like linear combinations of ξ^{2k}, ξ^{-2k}, $\xi^{2k}\ln\xi$, $\xi^{-2k}\ln\xi$ for $k > 0$. Except for a multiplying constant, these functions correspond to e^{-kz}, $z\,e^{-kz}$, e^{kz} and $z\,e^{kz}$, and consequently the DC requires that solutions should behave like $\xi^{2k}(A+B\ln\xi)$ near $\xi = 0$. For $k = 0$ the analogous result is $Z \sim A+B\ln\xi$. This proves the statement made in the previous section.

As $\epsilon \to 0$ the end point ξ_1 goes to ∞ along the ray $\arg\xi = -\frac{1}{4}\pi$, and therefore, we will want to make use of the asymptotic behaviour of $Z(\xi)$. There are two solutions which behave exponentially as $\xi \to \infty$:

$$S_1^{\pm}(\xi) \sim e^{\pm\xi}\xi^{-\frac{3}{2}}\sum_0^\infty b_n^{\pm}\xi^{-n} \quad (|\arg\xi| < \tfrac{1}{2}\pi), \tag{23}$$

where the coefficients can be determined by formally substituting in the differential equation. The exponentially growing solution, $S_1^+(\xi)$, is significant only near $\xi = \xi_1$, i.e. near $z = 0$. In fact, for small z, S_1^+ behaves approximately like $\exp[(i-1)z/(8\epsilon)^{\frac{1}{2}}]$, and thus corresponds to the boundary-layer solution. There are two other solutions which to first order behave asymptotically like solutions of the inviscid equation, Z_i. The form of the asymptotic series for these solutions depends on the values of γ and k, and the details are given in the appendix.

Let $S(\xi)$ be a solution which satisfies the DC and which grows less rapidly than $S_1^+(\xi)$ in the sector $|\arg\xi| < \frac{1}{2}\pi$. Since $S_1^-(\xi)$ vanishes faster than $Z_i(\xi)$, $S(\xi)$ behaves asymptotically like some solution of the inviscid equation (in $|\arg\xi| < \frac{1}{2}\pi$). If $S(\xi_1) \neq 0$ (it will be seen later that this is always the case for large ξ_1), $S(\xi)$ can be multiplied by a suitable constant to satisfy the boundary condition $Z(\xi_1) = d\sigma/Hk$. Furthermore, since $S_1^+(\xi)/\Theta S_1^+(\xi) \sim \xi^{-1}$, it is clear that one can add a multiple of the boundary-layer solution, $S_1^+(\xi)$, so as to satisfy $\Theta Z(\xi_1) = 0$ with only a small change in the value of S at ξ_1. Thus, a multiple of $S(\xi)$ will be a uniform approximation to the solution of the complete problem, $Z(\xi)$, in any fixed interval $0 \leqslant \xi \leqslant \xi'$ as $\epsilon \to 0$, and the asymptotic behaviour of $S(\xi)$ will determine the upper boundary condition for the inviscid problem. We will simplify the problem, therefore, by looking for $S(\xi)$ instead of $Z(\xi)$.

5. Series representation for $S(\xi)$

The solution $S(\xi)$ can be found by taking a suitable linear combination of series solutions relative to the regular singular point $\xi = 0$. One such solution satisfying the DC can be written in the form:

$$Z_1(\xi) = \sum_0^\infty a_n(\tfrac{1}{4}\xi^2)^{n+k}, \tag{24}$$

where $k \geqslant 0$ and the coefficients can be evaluated by substituting into the differential equation (19),

$$a_n = \frac{(n+k)^2-(n+k)+r}{n^2(n+2k)^2}\,a_{n-1} = \frac{(n+k-\frac{1}{2}-\gamma)(n+k-\frac{1}{2}+\gamma)}{n^2(n+2k)^2}\,a_{n-1} \tag{25}$$

$(n = 1, 2, \ldots)$. Setting aside temporarily the special case when $\gamma - k + \frac{1}{2}$ is a positive integer (for which the series in (24) terminates), we can write

$$a_n = Q\,\frac{\Gamma(n+k+\frac{1}{2}+\gamma)\,\Gamma(n+k+\frac{1}{2}-\gamma)}{2\Gamma^2(n+1)\,\Gamma^2(n+2k+1)}, \tag{26}$$

where
$$Q = Q(k,\gamma) = \frac{2\Gamma^2(2k+1)}{\Gamma(k+\frac{1}{2}+\gamma)\,\Gamma(k+\frac{1}{2}-\gamma)}, \tag{27}$$

and we have set $a_0 = 1$. Now, let $a(\zeta)$ be an analytic function which coincides with a_n when $\zeta = n$:

$$a(\zeta) = Q\,\frac{\Gamma(\zeta+k+\frac{1}{2}+\gamma)\,\Gamma(\zeta+k+\frac{1}{2}-\gamma)}{2\Gamma^2(\zeta+1)\,\Gamma^2(\zeta+2k+1)}. \tag{28}$$

Then, a second solution satisfying the DC can be written in the form

$$Z_2(\xi) = Z_1(\xi)\ln(\tfrac{1}{4}\xi^2) + \sum_0^\infty a'(n)(\tfrac{1}{4}\xi^2)^{n+k}$$
$$= \sum_0^\infty [\ln(\tfrac{1}{4}\xi^2) + a'(n)/a(n)]\, a_n(\tfrac{1}{4}\xi^2)^{n+k}. \tag{29}$$

Here

$$a'(n)/a(n) = \psi(n+k+\tfrac{1}{2}+\gamma) + \psi(n+k+\tfrac{1}{2}-\gamma) - 2\psi(n+1) - 2\psi(n+2k+1) \tag{30}$$

and $\psi(\zeta)$ is the Psi (or Digamma) function, defined by $\psi(\zeta) = \Gamma'(\zeta)/\Gamma(\zeta)$. The power series in the expressions for $Z_1(\xi)$ and $Z_2(\xi)$ are entire functions of ξ since the differential equation (19) has no singularities other than $\xi = \infty$ and the regular singularity at $\xi = 0$.

In §8 it will be shown that $Z_2(\xi)$ has the correct asymptotic behaviour required of $S(\xi)$. To show that apart from a multiplicative constant it is the only solution with the proper behaviour, we will now investigate the asymptotics of $Z_1(\xi)$.

Since the exponentially increasing solution dominates in the sector $|\arg|\xi < \tfrac{1}{2}\pi$, it is natural to compare the series for $Z_1(\xi)$ with a series of the same form which at ∞ behaves like $S_1^+(\xi)$. For this purpose one can employ the series for $\xi^{-1}I_1(\xi)$, where $I_1(\xi)$ is the modified Bessel function of the first kind of order one:

$$\xi^{-1}I_1(\xi) = \frac{1}{2}\sum_0^\infty (\tfrac{1}{4}\xi^2)^n/\Gamma(n+1)\,\Gamma(n+2)$$
$$\sim (2\pi)^{-\frac{1}{2}}\xi^{-\frac{3}{2}}e^{\xi} \quad \text{as} \quad \xi \to \infty\ (|\arg \xi| < \tfrac{1}{2}\pi).$$

Expression (23) shows that to first order the asymptotic behaviour of S_1^+ is independent of k. Hence, it will suffice to consider the case $k = 0$. For $k = 0$,

$$a_n = Q(0,\gamma)\frac{\Gamma(n+\frac{1}{2}+\gamma)\,\Gamma(n+\frac{1}{2}-\gamma)}{2\Gamma^4(n+1)}. \tag{31}$$

Making use of the asymptotic formula

$$\Gamma(z+a)/\Gamma(z+b) \sim z^{a-b}\{1+O(z^{-1})\},$$

we obtain for large n

$$a_n \sim Q(0,\gamma)[2\Gamma(n+1)\,\Gamma(n+2)]^{-1}\{1+O(n^{-1})\}. \tag{32}$$

By comparing with the series for $\xi^{-1}I_1(\xi)$, we conclude that for $\xi \to \infty$ ($|\arg \xi| < \tfrac{1}{2}\pi$)

$$Z_1(\xi) \sim Q(0,\gamma)\,\xi^{-1}I_1(\xi)$$
$$\sim Q(0,\gamma)(2\pi)^{-\frac{1}{2}}\xi^{-\frac{3}{2}}e^{\xi}, \tag{33}$$

[the higher order terms in the expression for a_n contribute a quantity which is $O(\xi^{-2}I_2(\xi))$, i.e. $O(\xi^{-\frac{5}{2}}e^{\xi})$]. Thus, the only solutions with the proper asymptotic behaviour are proportional to Z_2. It is convenient to take

$$S(\xi) = -\sum_0^\infty [\ln(\tfrac{1}{4}\xi^2) + a'(n)/a(n)]\, a_n(\tfrac{1}{4}\xi^2)^{n+k}. \tag{34}$$

6. Magnitude of the reflexion coefficient

From the form of the expression for $S(\xi)$ and what is known about its asymptotic behaviour, one can immediately deduce an important conclusion concerning the possibility of reflexion of waves. Let us consider the case $r > \frac{1}{4}$ $(\gamma = i\beta)$, i.e. the case of obliquely propagating waves. Since $\Gamma(\bar{\zeta}) = \overline{\Gamma(\zeta)}$ and $\psi(\bar{\zeta}) = \overline{\psi(\zeta)}$, it is clear from (26), (30) and (34) that $S(\xi)$ is real for real positive ξ. Furthermore, since $S(\xi)$ must behave like a solution to the inviscid equation for large ξ in the sector $|\arg \xi| < \frac{1}{2}\pi$, its asymptotic behaviour in that sector is given by

$$S(\xi) \sim \xi^{-1}(A\xi^{2i\beta} + B\xi^{-2i\beta}), \tag{35}$$

where A and B are complex constants which, in general, depend on β, k and δ. In order for the expression on the right-hand side of (35) to be real on the positive ξ-axis, A and B must be complex conjugates of each other, i.e.

$$S(\xi) \sim |A|\,\xi^{-1}(e^{i\phi}\xi^{2i\beta} + e^{-i\phi}\xi^{-2i\beta}), \tag{36}$$

where $\phi = \arg A$. Since real z corresponds to $\arg \xi = -\frac{1}{4}\pi$, let $\xi = se^{-i\frac{1}{4}\pi}$, and let us examine (36) for real positive s:

$$S \sim \text{const.}\, s^{-1}[e^{\frac{1}{2}\pi\beta + i\phi} s^{2i\beta} + e^{-(\frac{1}{2}\pi\beta + i\phi)} s^{-2i\beta}]. \tag{37}$$

Transforming back to the z variable, one obtains

$$S \sim \text{const.}\, e^{\frac{1}{2}z}\{\exp[\tfrac{1}{2}\pi\beta + i\phi - i\beta(z-\delta)] + \exp[-\tfrac{1}{2}\pi\beta - i\phi + i\beta(z-\delta)]\}, \tag{38}$$

or

$$S \sim \text{const}\, e^{\frac{1}{2}z}[e^{-i\beta z} + K_R e^{i\beta z}], \tag{39}$$

where

$$K_R = e^{-\pi\beta - 2i(\beta\delta + \phi)} \tag{40}$$

is the 'reflexion coefficient'. If the constant in (39) is determined so that the expression on the right-hand side is equal to $d\sigma/Hk$ when $z = 0$, we obtain an approximation to the solution of the problem in the IR,

$$Z(z) \sim (d\sigma/Hk)(1 + K_R)^{-1}(e^{-i\beta z} + K_R e^{i\beta z}). \tag{41}$$

The first term in (41) represents the 'incident wave', which has downward travelling phase and an upward energy flow, while the second term is the reflected wave, with reversed directions of phase and energy propagation. So far, only the magnitude of the reflexion coefficient, $|K_R| = e^{-\pi\beta}$, has been obtained; the phase constant, ϕ, will be determined later. It is important to note that although $|K_R|$ depends on β only, $\arg K_R$ depends on β, k and δ. Thus, if σ and k are kept constant and the viscosity, μ, tends to zero, the solution does *not* approach a limit at a fixed point z, since $\arg K_R$ varies with δ.

From (40) it is clear that the magnitude of K_R increases with decreasing β, i.e. with increasing wavelength in the vertical direction, and approaches 1 as $\beta \to 0$. Thus, when β is small there is almost total reflexion, and in the IR the solution looks like a standing wave (the limiting case $\beta = 0$ must be examined separately, however). As β increases $|K_R|$ decreases rapidly, becoming negligible (approximately 0·05) for $\beta = 1$. The radiation condition is, therefore, reasonably accurate for $\beta > 1$, but not for small β.

As we have remarked in the introduction, this situation is plausible, since the kinematic viscosity changes more rapidly per wavelength in the vertical direction as $\beta \to 0$, and can be expected to cause a reflexion. It is interesting to note, however, that a large change in the kinematic viscosity is required to produce a substantial reflexion, since even for $\beta = 1$, μ/ρ changes by a factor of about 400 in one vertical wavelength.

7. The case $(\gamma - k + \frac{1}{2})$ is a positive integer

The case where $\gamma - k - \frac{1}{2} = N$, a non-negative integer, can occur only for $r \leqslant 0$, i.e. for $\sigma \geqslant 1$ (in dimensional quantities, for $\sigma \geqslant (g/H)^{\frac{1}{2}}$). When r is negative the solution to the inviscid problem represents an oscillation which is already exponentially damped in z, and one expects the viscosity to have a relatively small effect. In this case, the series (24) will terminate after $N+1$ terms

$$Z_1(\xi) = \sum_0^N a_n (\tfrac{1}{4}\xi^2)^{n+k}, \tag{42}$$

and the asymptotic behaviour of $Z_1(\xi)$ is governed by the term of highest order

$$Z_1(\xi) \sim \text{const.}\, \xi^{2(N+k)} = \text{const.}\, \xi^{-1+2\gamma}, \tag{43}$$

i.e. $Z_1(\xi)$ behaves like a solution to the inviscid problem. A second solution in this case is given by

$$Z_3(\xi) = Z_1(\xi) \ln(\tfrac{1}{4}\xi^2) + \sum_0^\infty d_n (\tfrac{1}{4}\xi^2), \tag{44}$$

and for $n > N$, the coefficients d_n satisfy the recursion relation (25). Therefore, $Z_3(\xi)$ behaves asymptotically like $\xi^{-1} I_1(\xi)$. Consequently, apart from a multiplying constant, $Z_1(\xi)$ is the only solution with the required asymptotic behaviour, and we can take

$$S(\xi) = \text{const.} \sum_0^N a_n (\tfrac{1}{4}\xi^2)^{n+k}. \tag{45}$$

For $\epsilon \to 0$ (i.e. $\xi_1 \to \infty$) the boundary condition $Z(\xi_1) = d\sigma/Hk$ implies

$$Z(\xi) = (d\sigma/Hk)(\xi/\xi_1)^{-1+2\gamma} + O(\xi_1^{-2}), \tag{46}$$

i.e.

$$Z(z) = (d\sigma/Hk)\, e^{-(\gamma-\frac{1}{2})z} + O(\epsilon), \tag{47}$$

where the approximation holds uniformly for all z (to obtain an approximation for $Z'(z)$ one must add on a boundary-layer solution). In the special case $N = 0$, the series consists of one term, which is, in fact, just the solution to the inviscid problem. Consequently, if $\gamma - k - \frac{1}{2} = N$, the solution to the inviscid problem is a good approximation to the solution of the complete problem.

8. Integral representation and asymptotic behaviour of $S(\xi)$

Previously a heuristic derivation of the magnitude of the reflexion coefficient, $K_R = e^{-\pi\beta}$, was given. Now, this result will be proved by computing the asymptotic behaviour of $S(\xi)$. In the process the previously unknown phase constant, ϕ, will be determined, and at the same time the cases $r < \frac{1}{4}$ and $r = \frac{1}{4}$ will be treated.

Throughout this section it will be assumed that $\gamma - k + \frac{1}{2}$ is not a positive integer, so that $S(\xi)$ is given by the series (34) and the coefficients a_n by (26), or by (28) with $\zeta = n$. The asymptotic behaviour of $S(\xi)$ for large ξ in the sector $|\arg \xi| < \frac{1}{2}\pi$ will be obtained by first transforming the series (34) into an integral by means of the residue theorem, and then estimating this integral (see e.g. Ford 1960).

The presence of $a'(n)$ in the coefficients of the series (34) suggests the use of an integrand with poles of second order at the integers. Let us consider an integral of the form

$$I_{NM} = -(1/2\pi i) \int_{C_{NM}} F(\zeta, \xi) G(\zeta) d\zeta, \tag{48}$$

where
$$F(\zeta, \xi) = a(\zeta) e^{2(\zeta+k) \ln \frac{1}{2}\xi}, \quad G(\zeta) = \pi^2 \csc^2 \pi\zeta \tag{49}$$

and the path of integration will be specified later. Since $\gamma - k + \frac{1}{2} \neq$ integer, $a(\zeta)$ is regular at $\zeta = n$, and since $G(\zeta) \sim (\zeta - n)^{-2} + \frac{1}{3}\pi^2 \ldots$ near $\zeta = n$, the contribution of the integral from the residue at $\zeta = n$ ($n = 0, 1, 2, \ldots$) is $-dF(n, \xi)/d\zeta$, i.e.

$$-[\ln \tfrac{1}{4}\xi^2 + a'(n)/a(n)] a(n) (\tfrac{1}{4}\xi^2)^{n+k}. \tag{50}$$

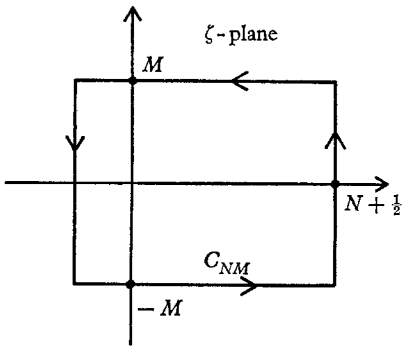

FIGURE 1. Contour of integration.

Consequently, to sum the series (34), let us take for C_{NM} a rectangular contour (see figure 1) with the right-hand vertical portion lying along $\operatorname{Re} \zeta = N + \frac{1}{2}$ (N = integer), the two horizontal portions along $\operatorname{Im} \zeta = \pm M$, and the left-hand vertical portion being somewhat to the left of the imaginary ζ-axis. Then, I_{NM} is equal to the sum of the first $N+1$ terms of (34), plus the contribution due to whatever other singularities are enclosed within C_{NM}. To obtain an integral representation for $S(\zeta)$, we will take the limit as $N \to \infty$.

From the expression (28) for $a(\zeta)$ and the asymptotic formula for the Gamma function, one obtains

$$a(\zeta) \sim (4\pi)^{-1} Q \zeta^{-2(k+1)} e^{-2\zeta(\ln \zeta - 1)} \tag{51}$$

for large ζ, $|\arg \zeta| < \pi$, and

$$F(\zeta, \xi) \sim \text{const.}\, \xi^{2k} \zeta^{-2k-2} e^{-2\zeta(\ln 2\zeta/\xi - 1)}. \tag{52}$$

Thus,

$$|F(\zeta, \xi)| \sim \text{const.}\, |\xi|^{2k} |\zeta|^{-2k-2} \exp\{-2 \operatorname{Re} \zeta (\ln |2\zeta/\xi| - 1) + 2 \operatorname{Im} \zeta (\arg \zeta - \arg \xi)\}. \tag{53}$$

Since $G(\zeta)$ is bounded on $\operatorname{Re}\zeta = N+\frac{1}{2}$, it is clear that for any fixed ξ the contribution to I_{NM} from the right-hand side of C_{NM} goes to zero as $N\to\infty$. Consequently, letting C_M be the path shown in figure 2, and I_M the integral over C_M, we can see that $I_M - S(\xi) =$ contribution due to singularities other than $\zeta = n$.

We will now show that the contribution from the horizontal portions of C_M tends to zero as $M\to\infty$. To see this, let us note that

$$G(\zeta) \sim 4e^{-2\pi|\operatorname{Im}\zeta|} \tag{54}$$

as $|\operatorname{Im}\xi|\to\infty$. Consequently, for large M,

$$|F(\zeta,\xi)G(\zeta)| \sim \text{const.}\,|\xi|^{2k}|\zeta|^{-2k-2} \times\exp\{-2\operatorname{Re}\zeta(\ln|2\zeta/\xi|-1)+2\operatorname{Im}\zeta(\arg\zeta-\arg\xi\mp\pi)\}, \tag{55}$$

where the upper and lower signs apply only to the upper and lower sides of C_M respectively. Now, if ξ is any fixed number with $-\frac{1}{2}\pi+\eta < \arg\xi < \frac{1}{2}\pi-\eta$, where η is an arbitrarily small positive quantity, it is clear that $\operatorname{Im}\zeta(\arg\xi-\arg\xi\mp\pi)$ is always negative on the upper and lower sides of C for sufficiently large M. Consequently, for any fixed ξ with $|\arg\xi| < \frac{1}{2}\pi$, the contribution from the horizontal parts of C_M tends to zero exponentially as $M\to\infty$, and C_M can be replaced by a vertical straight line (traversed downward) to the left of the imaginary ζ-axis.

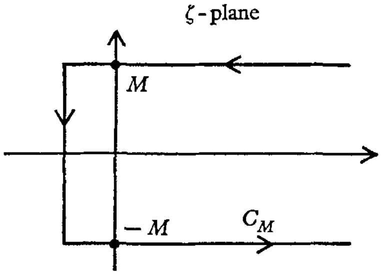

FIGURE 2. Contour of integration.

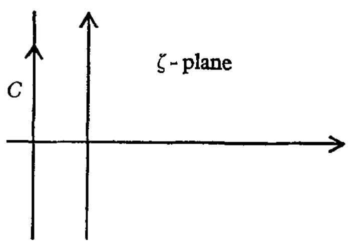

FIGURE 3. Contour of integration.

The function $a(\zeta)$ has simple poles at $\zeta = \zeta_\pm = -(\frac{1}{2}+k)\pm\gamma$ and at $\xi = \xi_\pm - n$ $(n = 1,2,\ldots)$ if $r \neq \frac{1}{4}$, and poles of second order at $\zeta = -(\frac{1}{2}+k+n)$ if $r = \frac{1}{4}(\gamma=0)$. In view of the restriction $\gamma-k+\frac{1}{2}\neq N$, all these poles are distinct from the integers. Letting C be a vertical straight line traversed in the upward direction (see figure 3) and not passing through any of the poles, we obtain

$$S(\xi) = (1/2\pi i)\int_C F(\zeta,\xi)G(\zeta)\,d\zeta - \sum_i R_i, \tag{56}$$

where R_i denote the residues of $F(\zeta,\xi)G(\zeta)$ at those poles of $a(\zeta)$ which are to the right of C. If $\operatorname{Re}\gamma < \frac{1}{2}+k$ (in particular, if $\gamma = i\beta$) all the poles are to the left of the imaginary axis, and if C lies between $\zeta_\pm$ and the imaginary axis, $S(\xi)$ is given by the integral

$$S(\xi) = (1/2\pi i)\int_C \pi^2\csc^2\pi\zeta\, a(\zeta)\, e^{2(\zeta+k)\ln\frac{1}{2}\xi}\,d\zeta. \tag{57}$$

Case 1. $\gamma = i\beta \quad (r > \frac{1}{4})$

Let C be the contour $\operatorname{Re}\zeta = -(1+k)$. Then, the pair of poles $\zeta = \zeta_{\pm}$ lie to the right of C, while all the rest of the poles of $a(\zeta)$ are to the left of C. Letting I_C denote the integral in (57), we can write

$$S(\xi) = I_C - (R_+ + R_-), \tag{58}$$

where R_+ and R_- denote the residues of $F(\zeta, \xi)G(\xi)$ at ζ_+ and ζ_-. First we will estimate the integral I_C. On C

$$|e^{2(\zeta+k)\ln\frac{1}{2}\xi}| = \tfrac{1}{2}|\xi|^{-2}e^{-2\operatorname{Im}\zeta(\arg\xi)}. \tag{59}$$

Letting $|\xi| \to \infty$ with $\arg\xi$ fixed, we can see that the integrand in I_C is $O(\xi^{-2})$. Furthermore, since the integral taken along C converges absolutely, this shows that

$$I_C = O(\xi^{-2}) \tag{60}$$

as $\xi \to \infty$ with $|\arg\xi| < \frac{1}{2}\pi$.

The residue R_+ at $\zeta_+ = -(\frac{1}{2}+k)+i\beta$ is given by

$$R_+ = \frac{\pi^2 Q \csc^2\pi(-\frac{1}{2}-k+i\beta)\,\Gamma(2i\beta)}{2\Gamma^2(\frac{1}{2}-k+i\beta)\,\Gamma^2(\frac{1}{2}+k+i\beta)}\,(\tfrac{1}{2}\xi)^{-1+2i\beta}. \tag{61}$$

Making use of the identity

$$\Gamma(z)\,\Gamma(1-z) = \pi\csc\pi z,$$

we can write this expression in the form

$$R_+ = -A\xi^{-1+2i\beta}, \tag{62}$$

where $A = A(k,\beta)$ is given by

$$A = -Q2^{-2i\beta}\,\Gamma(2i\beta)\,\frac{\Gamma^2(\frac{1}{2}+k-i\beta)}{\Gamma^2(\frac{1}{2}+k+i\beta)}. \tag{63}$$

It is evident from (27) that Q is positive. Thus,

$$A = Q|\Gamma(2i\beta)|\,e^{i\phi}, \tag{64}$$

where

$$\phi = \phi(k,\beta) = \pi - 2\beta\ln 2 + \arg\Gamma(2i\beta) - 4\arg\Gamma(\tfrac{1}{2}+k+i\beta). \tag{65}$$

The residue at $\zeta_- = -(\frac{1}{2}+k)-i\beta$ can be obtained from the above formulas simply by changing the sign of β, and it is evident that $A(k,-\beta) = \overline{A(k,\beta)}$. Since the contribution due to I_C is small compared to $(R_+ + R_-)$, we obtain

$$\begin{aligned} S(\xi) &\sim \xi^{-1}(A\xi^{2i\beta} + \bar{A}\xi^{-2i\beta}) \\ &\sim |A|\,\xi^{-1}(e^{i\phi}\xi^{2i\beta} + e^{i\phi}\xi^{-2i\beta}) \end{aligned} \tag{66}$$

for large $|\xi|$ in the sectoı $|\arg\xi| < \frac{1}{2}\pi$. This expression is identical with (36) and this proves the results developed previously, with the phase constant ϕ now being known from (65). Since $\Gamma(z)$ cannot vanish ($1/\Gamma(z)$ being an entire function), A cannot vanish, and its magnitude, therefore, is unimportant since the solution must be normalized to satisfy the appropriate boundary condition. It should be noted that more terms in the asymptotic representation for $S(\xi)$ can be obtained by moving the path C to the left and evaluating the residues at $\zeta = \zeta_{\pm} - n$, or directly from the formal asymptotic series (see appendix).

Case 2. $\gamma = \alpha \quad (r < \frac{1}{4})$

It is clear that the main contribution will come from the pole of $a(\zeta)$ which is farthest to the right, i.e. from $\zeta_+ = -(\frac{1}{2}+k)+\alpha$. Choosing the path of integration C to be the straight line $\operatorname{Re}\zeta = -(1+k)+\alpha$, and using the same procedure as in the previous case, we obtain

$$S(\xi) \sim \text{const.}\, \xi^{-1+2\alpha}[1+O(\xi^{-1})] \tag{67}$$

for $\xi \to \infty$, $|\arg\xi| < \frac{1}{2}\pi$, i.e.

$$Z(z) \sim (d\sigma/Hk)\, e^{(\frac{1}{2}-\alpha)z} \tag{68}$$

in the IR. Thus, as the viscosity goes to zero, the solution of the viscous problem approaches that solution of the inviscid problem which has finite kinetic energy. If the solution is required only in some fixed interval $z_1 < z < z_2$, and if μ is sufficiently small, the viscous problem can be replaced by the inviscid one. Of course, when $\alpha < \frac{1}{2}$ the effect of viscosity is important in attenuating the solution at large altitude. For $\alpha > \frac{1}{2}$, when the solution to the inviscid problem decays with increase in z, the effect of viscosity can be expected to be negligible.

Case 3. $\gamma = 0 \quad (r = \frac{1}{4})$

The poles ζ_+ and ζ_- now coalesce into a double pole at $\zeta = -(k+\frac{1}{2})$, and one can take the line $\operatorname{Re}\zeta = -(1+k)$ for the contour C. The contribution of I_C is again $O(\xi^{-2})$, and evaluation of the residue yields (for $|\arg\xi| < \frac{1}{2}\pi$)

$$\begin{aligned} S(\xi) &\sim \xi^{-1}[\ln(\tfrac{1}{4}\xi^2) - 2\pi\tan k\pi + 2\psi(1) - 2\psi(\tfrac{1}{2}-k) - 2\psi(\tfrac{1}{2}+k)] \\ &\sim \xi^{-1}[\ln(\tfrac{1}{4}\xi^2) + 2\psi(1) + 4\psi(\tfrac{1}{2}+k)], \end{aligned} \tag{69}$$

where we have made use of the relation $\psi(z) - \psi(1-z) + \pi\cot\pi z = 0$. Thus,

$$Z(z) \sim \text{const.}\, e^{\frac{1}{2}z}(z+q), \tag{70}$$

where

$$q = \tfrac{1}{2}i\pi - \delta + 2[2\psi(\tfrac{1}{2}+k) - \psi(1) + \ln 2]. \tag{71}$$

As $\mu \to 0$ ($\epsilon \to 0$), $\delta \to \infty$ and for any fixed interval in the z solution $Z(z)$ tends to a constant multiple of $e^{\frac{1}{2}z}$. However, unless the viscosity μ is very small the $ze^{\frac{1}{2}z}$ term in (69) will not be negligible (see §10).

9. Computations

To obtain an estimate on the extent of the IR and the TR, and to get some idea of the behaviour of solutions outside the IR, a number of solutions were computed for various values of k and r. Some typical results are shown in figures 4–8.

Because of the presence of an exponentially growing component in solutions of the differential equation (19), $S(s)$ was computed by using the series representation

$$S(s) = \sum_0^\infty [\tfrac{1}{2}i\pi - \ln(\tfrac{1}{4}s^2) - a'(n)/a(n)]\, a_n(-\tfrac{1}{4}is^2)^{n+k}$$

with $a'(n)/a(n)$ given by (30) and $s = \exp[\frac{1}{2}(\delta - z)]$. The horizontal velocity $U = -\frac{1}{2}\Theta S(s)$ was obtained from the corresponding series. It was necessary to use double precision because the maximum term of the series grows exponentially with s, and each solution was extended to values of s for which the behaviour of $s|S|$ agreed well with the first term of its asymptotic expansion. This limited the range of β to less than about 2 ($r < 4{\cdot}25$).

To obtain $a'(n)/a(n)$ to the required degree of accuracy the expression (30) has to be rearranged. Making use of the functional relation $\psi(z+1) = \psi(z)+1/z$, one can write

$$\begin{aligned}\psi(n+k+\tfrac{1}{2}+\gamma)+\psi(n+k+\tfrac{1}{2}-\gamma)-2\psi(n+1) &= \psi(k+\tfrac{1}{2}+\gamma)\\ &\quad +\psi(k+\tfrac{1}{2}-\gamma)-2\psi(1)+2\sum_{1}^{n}\frac{2(m+k)-1}{(m+k)^2-(m+k)+r}\end{aligned} \tag{72}$$

and

$$\begin{aligned}\frac{a'(n)}{a(n)} = \sum_{1}^{n}\frac{2(m+k)-1}{(m+k)^2-(m+k)+r}-2\sum_{1}^{n}\frac{1}{m}-2\sum_{1}^{n}\frac{1}{m+2k}\\ +\psi(k+\tfrac{1}{2}+\gamma)+\psi(k+\tfrac{1}{2}-\gamma).\end{aligned} \tag{73}$$

The asymptotic expression, $\psi(z) = \ln z - 1/z + \ldots$, shows that the left-hand side of (72) approaches zero as $n \to \infty$, and thus one obtains an evaluation of $\psi(k+\frac{1}{2}+\gamma)+\psi(k+\frac{1}{2}-\gamma)$. Finally, letting

$$T(n,k) = \sum_{1}^{n}\frac{1-2k}{(m+k)^2-(m+k)+r}+2\sum_{1}^{n}\frac{k(1-k)-r}{m[(m+k)^2-(m+k)+r]}, \tag{74}$$

one can show that

$$\left.\begin{aligned}\frac{a'(n)}{a(n)} &= T(n,k)-T(\infty,k)-2\sum_{1}^{n}\frac{1}{m+2k}-2\psi(1+2k) \quad (n = 1,2,\ldots)\\ &= -T(\infty,k)-2\psi(1+2k) \quad (n = 0).\end{aligned}\right\} \tag{75}$$

Since double precision had to be used, the constant $T(\infty,k)$ was computed to 25 decimal places by means of the Euler–McLaurin formula.

10. Discussion of results

In discussing the computed solutions it will be convenient to distinguish three ranges of the parameter r, which, in dimensional quantities, is equivalent to $(gH/V^2-k^2H^2)$, where $V = \sigma/k$ is the horizontal phase velocity. The three ranges are: $r \leqslant 0$ (for which the dimensional $\sigma \geqslant \sqrt{(gH)}$), $0 < r < \frac{1}{4}$, and $r > \frac{1}{4}$. The transition value, $r = \frac{1}{4}$ will be considered separately. For long horizontal wavelengths, $r = \frac{1}{4}$ corresponds to $V = 2\sqrt{(gH)}$.

For $r < 0$ the solutions of the inviscid problem decay exponentially with z and viscosity is not expected to have an appreciable effect. This was already noted for the special case $\gamma - k + \frac{1}{2} = N$. In figure 4 solutions for $r = -0{\cdot}75$ and two different values of k are shown. The curves show the non-dimensional horizontal velocity amplitude U plotted against s, i.e. against $[\rho/\rho(0)\epsilon]^{\frac{1}{2}}$. It can be seen that the graphs are almost indistinguishable from straight lines which correspond to solutions of the inviscid problem.

For the remaining cases ($r > 0$) the solutions of the inviscid problem grow exponentially with z and viscosity plays an important part. Some results for

$r = 0{\cdot}16, 0{\cdot}25, 0{\cdot}5, 1{\cdot}25$ and $2{\cdot}5$ are shown in figures 5–8. For $0 < r < \frac{1}{4}$ it is convenient to plot $|U|$ against $\ln s$, and for $r \geqslant \frac{1}{4}$, $s|U|$ against $\ln s$, since this clearly displays the asymptotic behaviour. When s is large the solutions of the viscous problem behave like appropriate solutions of the inviscid problem, which are given by the asymptotic formulas derived in §8. This range of s, therefore, corresponds to the IR. For sufficiently small s the solution $S(s)$ can be approximated

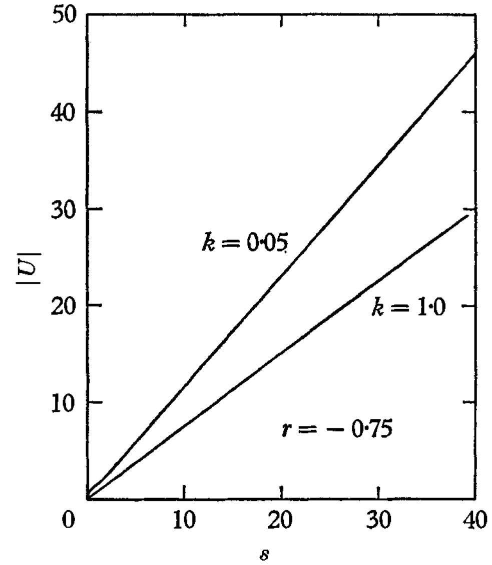

FIGURE 4. Solution of the viscous problem for $r = -0{\cdot}75$ and $k = 0{\cdot}05, 1{\cdot}0$. U = horizontal velocity amplitude$/(gH)^{\frac{1}{2}}$, $s = [\rho/\rho(0)\epsilon]^{\frac{1}{2}}$.

by the term $-\ln(\frac{1}{4}s^2)(-\frac{1}{4}is^2)^k$, and the region where this approximation is valid corresponds to the VR. The TR lies in between these two. Of course, these designations are not precisely defined, but there is no difficulty in identifying the different regions on the graphs. It should be noted that the solutions for $k = 0$ should be interpreted as approximations to solutions for small k, the approximations being valid except in the neighbourhood of $s = 0$. One cannot take the limit as $k \to 0$ (with μ fixed) since this necessitates $\sigma \to 0$, which implies $\epsilon \to \infty$. However, for reasonable values of μ and H one can still consider oscillations with periods of about 10^6 hours.

In order to translate the result into physical terms one must choose values for μ and H. We will take μ to be 2×10^{-5} kg/sec, which corresponds to the value in the atmosphere at about 120 km.† Then, the value of the small parameter ϵ is approximately 5×10^{-11} for oscillations with a period of 1 h, and the corresponding value of δ is 23·7. Letting the scale height be 7 km, we obtain approximately 3 min for the characteristic period $2\pi\sqrt{(H/g)}$ and 265 m/sec for the characteristic velocity $\sqrt{(gH)}$. It seems likely that most of our conclusions for low-frequency oscillations will apply qualitatively to the case of an isothermal atmosphere. However, quantitatively there will be some differences, since the Brunt–Väisälä

† *U.S. Standard Atmosphere*, 1962 Washington, D.C., U.S. Government Printing Office.

frequency is lowered and the characteristic period will be about 5 min for the same value of H.

In the solutions which were computed ($r \leqslant 4{\cdot}25$, $0 \leqslant k \leqslant 1$), the IR generally appeared to end a little below the point where $\rho(z) = \epsilon$. For example, for $r = 0{\cdot}16$ and a period of 12 h the IR is roughly $z < 20$ (scale heights), i.e. below 140 km.

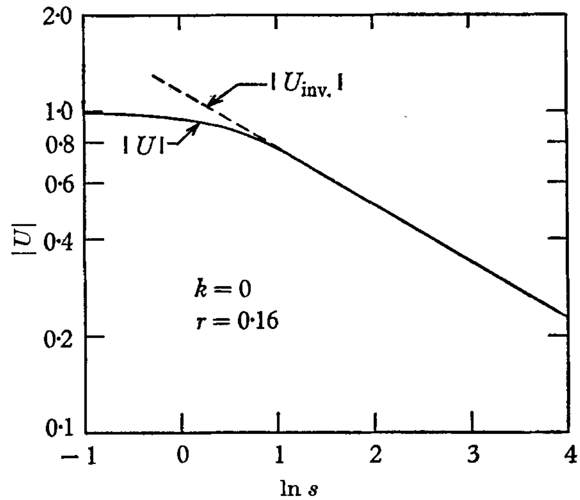

FIGURE 5. Solutions of the viscous problem for $r = 0{\cdot}16$, $k = 0$. U = horizontal velocity amplitude$/(gH)^{\frac{1}{2}}$, $s = [\rho/\rho(0)\epsilon]^{\frac{1}{2}}$. $U_{\text{inv.}}$ = solution of inviscid problem.

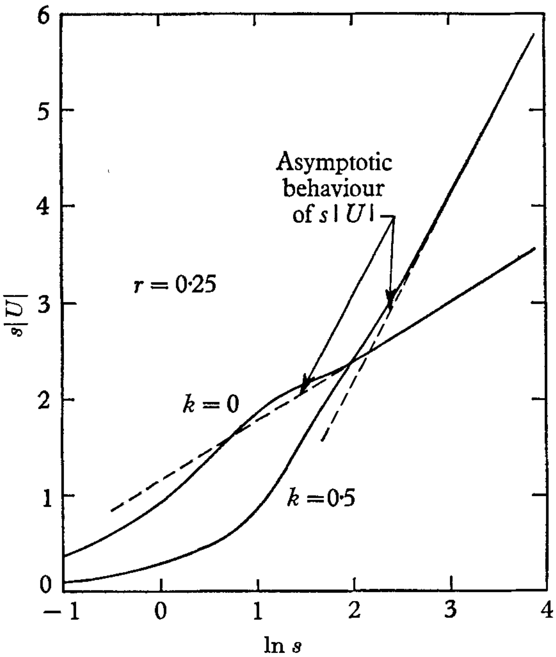

FIGURE 6. Solutions of the viscous problem for $r = 0{\cdot}25$ and $k = 0, 0{\cdot}5$. U = horizontal velocity amplitude$/(gH)^{\frac{1}{2}}$, $s = [\rho/\rho(0)\epsilon]^{\frac{1}{2}}$.

For $r = 2{\cdot}5$ and a period of 12 h, it is roughly $z < 17$, i.e. below about 120 km. The IR shrinks as μ and r increase, and as k decreases. The TR was found to be fairly thin, its thickness varying from 5 to 8 scale heights. For $k = 0{\cdot}5$ and $k = 1{\cdot}0$ the velocity has its maximum value in the TR and decays quite rapidly from then on. For small values of k, however, there is a velocity maximum near $s = 0$, i.e. high up in the VR.

Figure 6 shows solutions for the transition value $r = \frac{1}{4}$ with $k = 0$ and $k = 0{\cdot}5$. The asymptotic behaviour is known from (69) to (71). The values of δ for $k = 0{\cdot}5$

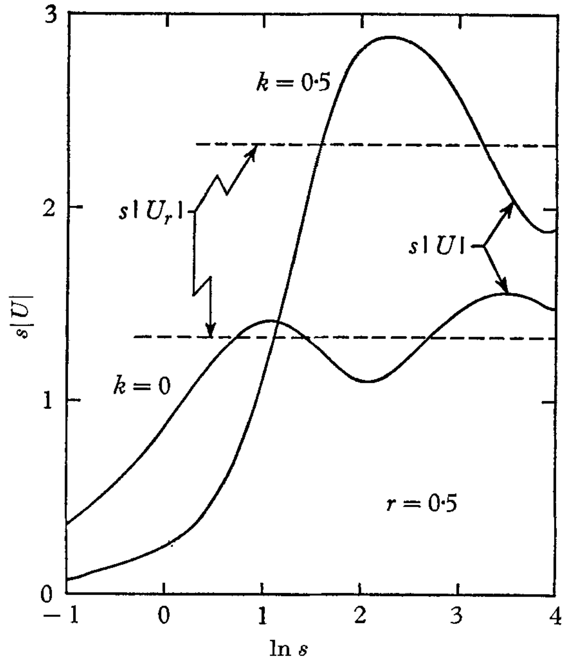

FIGURE 7. Solutions of the viscous problem for $r = 0{\cdot}5$ and $k = 0, 0{\cdot}5$. U = horizontal velocity amplitude$/(gH)^{\frac{1}{2}}$, $s = [\rho/\rho(0)\epsilon]^{\frac{1}{2}}$. U_r = solution of inviscid problem satisfying radiation condition.

and for a period of 12 h are 27 and 21·4 respectively. Consequently, μ is not small enough for the $ze^{\frac{1}{2}z}$ term in (70) to be negligible in the IR. Solutions for $r = 0{\cdot}5$ and $r = 2{\cdot}5$, i.e. for vertical wave-numbers $\beta = 0{\cdot}5$ and $\beta = 1{\cdot}5$, are shown in figures 7 and 8. To make comparison with solutions of the inviscid problem easier we have plotted the quantity $s|U|$. For $\beta = 1{\cdot}5$ (and both $k = 0$ and $k = 0{\cdot}5$) the reflexion is small and $s|U|$ in the IR is close to a horizontal line which represents the solution of the inviscid problem satisfying the radiation condition. For $\beta = 0{\cdot}5$, however, the reflexion is considerable and the solution in the IR departs noticeably from the solution obtained by using the radiation condition.

Linearization

A rough idea of the amplitude of the boundary oscillation, d/H, for which the linearization is valid can be obtained by examining the ratio of the density perturbation to the equilibrium density, $\tilde{\rho}/\rho$. For small k the growth of this ratio with z is much greater than the growth of the velocity. From (4) one obtains

$$\tilde{\rho}/\rho = -(k/\sigma)Z, \tag{76}$$

and since $Z(0) = d\sigma/Hk$, $|\tilde{\rho}(0)/\rho(0)| = d/H$. For oscillations with a period of 12 h it was found that the density ratio at $z = 0$ must be less than about 10^{-6} in order for it to remain less than 0·2. For shorter periods and $k = 0{\cdot}5$, d/H has to be less than about 10^{-5}. Thus, it might be expected that non-linear effects may still be important in problems such as the problem of atmospheric tides.

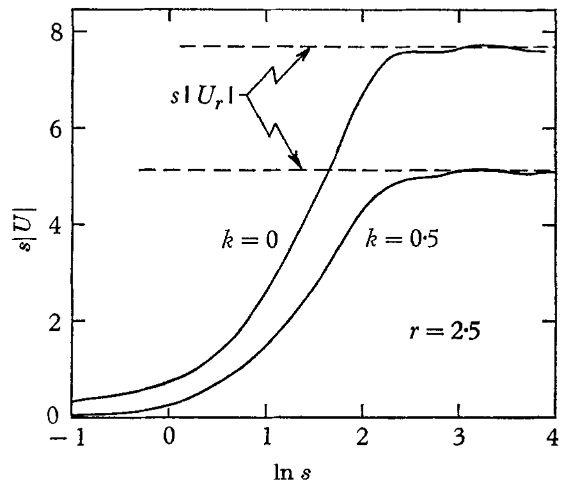

FIGURE 8. Solutions of the viscous problem for $r = 2{\cdot}5$ and $k = 0,\ 0{\cdot}5$. U = horizontal velocity amplitude$/(gH)^{\frac{1}{2}}$, $s = [\rho/\rho(0)\epsilon]^{\frac{1}{2}}$. U_r = solution of inviscid problem satisfying radiation condition.

11. Table of symbols

x, z Horizontal and vertical co-ordinates
t Time
u, w Horizontal and vertical velocity components
ρ Undisturbed density (function of z only)
$\tilde{p}, \tilde{\rho}$ Pressure and density perturbations
k Horizontal wave-number
σ Frequency
Ψ Stream function
Z Stream function amplitude, $\Psi = Z e^{i(kx-\sigma t)}$
The quantities x^*, z^*, ρ^*, k^*, σ^*, Z^* are dimensionless quantities defined in (8). The * is generally omitted after formula (9)
g Acceleration of gravity
H $= (-\rho'/\rho)^{-1}$, density scale height
d Amplitude of oscillation of the lower boundary
μ Dynamic viscosity coefficient
ϵ $= \mu[4\rho(0)H^2\sigma]^{-1}$, small dimensionless parameter

δ $= -\ln \epsilon$, dimensionless parameter
r $= (k^*/\sigma^*)^2 - k^{*2}$, dimensionless constant
γ $= (\frac{1}{4} - r^2)^{\frac{1}{2}}$
α $= \gamma$, if $r < \frac{1}{4}$
β $= (r^2 - \frac{1}{4})^{\frac{1}{2}}$ if $r > \frac{1}{4}$, dimensionless vertical wave-number
λ $= 2\pi H/\beta$, vertical wavelength
ξ $= \epsilon^{-\frac{1}{2}} \exp(-\frac{1}{2}z^* + \frac{1}{4}i\pi)$
s $= \epsilon^{-\frac{1}{2}} \exp(\frac{1}{2}z^*)$
Θ $= \xi d/d\xi$, differential operator
Γ Euler's Gamma Function
ψ Psi or Digamma Function
K_R Reflexion coefficient
ϕ Phase constant defined in (65)
U Dimensionless horizontal velocity amplitude, referred to $(gH)^{\frac{1}{2}}$

Appendix

Let us look for formal asymptotic series of the form

$$Z^{\pm}(\xi) \sim \xi^{-1\pm 2\gamma} \sum_0^{\infty} c_n^{\pm} \xi^{-2n}. \tag{A1}$$

Substituting in the differential equation (19) yields the recursion relation

$$n(n \mp 2\gamma) c_n^{\pm} = 4(n - \tfrac{1}{2} \mp \gamma - k)^2 (n - \tfrac{1}{2} \mp \gamma + k)^2 c_{n-1}^{\pm}. \tag{A2}$$

Two infinite asymptotic series of the above type can always be determined if the following condition is satisfied:

(A) $2\gamma \neq n$ for $n = 0, 1, 2, \ldots$. There will also be cases where one of the two series will terminate, and will thus represent a solution of (19) exactly. If

(B) condition A holds and $\pm k = N - \gamma + \frac{1}{2}$ for some non-negative integer N, (where the plus sign is taken when the right-hand side is positive, and the minus sign if it is negative), then the series for $Z^+(\xi)$ terminates:

$$Z^+(\xi) = \xi^{-1+2\gamma} \sum_0^N c_n^+ \xi^{-2n} = \xi^{2(N \mp k)} \sum_0^N c_n^+ \xi^{-2n}. \tag{A3}$$

The solution corresponding to the lower sign satisfies the DC and is equivalent to the solution obtained in §7. The other solution does not satisfy the DC. If

(C) $2\gamma = N + 1$ for some non-negative integer N, and $2k = N - 2M$ for some positive integer $M \leqslant \frac{1}{2}N$, the series for $Z^+(\xi)$ terminates again:

$$Z^+(\xi) = \xi^{-1+2\gamma} \sum_0^M c_n^+ \xi^{-2n} = \xi^{2(k+M)} \sum_0^M c_n^+ \xi^{-2n}. \tag{A4}$$

This solution is also equivalent to the solution obtained in §7. If

(D) condition A holds and $k = N + \gamma + \frac{1}{2}$ for some non-negative integer N, then the series for $Z^-(\xi)$ terminates:

$$Z^-(\xi) = \xi^{-1-2\gamma} \sum_0^N c_n^- \xi^{-2n} = \xi^{2(N-k)} \sum_0^N c_n^- \xi^{-2n}. \tag{A5}$$

In all other cases the asymptotic series for $Z^+(\xi)$ will contain a logarithmic term. If

(E) $\gamma = 0$, or if

(F) $2\gamma = N+1$ for some non-negative integer n, and $2k$ is not a positive integer, the asymptotic expression for $Z^+(\xi)$ has the form

$$Z^+(\xi) \sim Z^-(\xi) \ln \xi + \xi^{-1+2\gamma} \sum_0^\infty c'_n \xi^{-2n}, \tag{A6}$$

while the series for $Z^-(\xi)$ has the form (A1). The last series will terminate if

(G) $2\gamma = N+1$ for some non-negative integer N, and $k-1-\frac{1}{2}N = M$, a non-negative integer,

$$Z^-(\xi) = \xi^{-1-2\gamma} \sum_0^M c_n^- \xi^{-2n} = \xi^{2(M-k)} \sum_0^M c_n^- \xi^{-2n}. \tag{A7}$$

In all the cases solutions behave asymptotically like solutions of the inviscid equation.

I would like to express my appreciation to Mrs S. Hargreaves, who programmed all the computations. This research was done while the author was on leave at the National Center for Atmospheric Research, Boulder, Colorado, and was supported in part by the National Science Foundation, NSF GP-1804.

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